

Proposed Nomenclature for Linear Viscoelastic Behavior*

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I. INTRODUCTION

A systematic standardized nomenclature is very desirable to facilitate communication among physicists working in the same field or in associated fields; such a nomenclature gives names and symbols to measurable related quantities. After the fundamentals of a branch of physics are fairly well established, it is then possible to draw up a logical complete nomenclature system. Unfortunately, by that time the field is usually burdened with several systems which are not generally accepted and which are not completely logical; these systems may or may not be related to nomenclature systems in associated branches of physics.

Ideal nomenclature systems which are prepared subsequent to the understanding of the relationships between measurable quantities in a branch of physics are often proposed. They may represent radical departures from more or less established usage; under these circumstances the proposals do not gain general acceptance. A practical improvement on a haphazard nonstandardized nomenclature can only be gained by adhering as closely as possible to existing usage. In this report an attempt has been made to devise a nomenclature following this principle.

When certain materials are deformed such that the stored elastic energy and the rate of dissipation of mechanical energy are sufficiently small, then the mechanical behavior of these materials approximates to the idealized behavior known as *linear viscoelastic behavior*, which may be defined in general by equations such as (1) and (2) given

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below. This report deals with a proposed nomenclature for primary measurable quantities and for related quantities applicable to *isotropic* materials manifesting (approximate) linear viscoelastic behavior. While this nomenclature is planned primarily in order to systematize the somewhat extensive literature on the behavior of amorphous polymeric systems, it is also adaptable, perhaps with some modifications, to other materials such as inorganic glasses and metals. The *proposed nomenclature* is given in Table II, which contains *primary measurable quantities*. In Table III are given certain *auxiliary quantities*, namely, limiting values and spectra. No attempt has been made to make this nomenclature consistent with the nomenclature for dielectric behavior, since in practice only one of the four primary methods of presenting data considered in this report is used in the specification of dielectric behavior. Furthermore, no attempt has been made to prepare a *complete* system of nomenclature. For materials manifesting linear viscoelastic behavior, many different methods of specifying behavior are possible; in the present proposed nomenclature a minimum number of measurable and related quantities are defined. The text of this report gives the basis for the nomenclature, and some of the relations between the quantities specified in Tables II and III. For the reasons given above, generally accepted usage has been followed as much as possible.

II. STEP-FUNCTION AND DYNAMIC MODULI AND COMPLIANCES

The linear viscoelastic behavior of a material (at a given temperature) is generally specified by the response of the material in certain types of experiment. While any reasonable type of experiment is theoretically acceptable, in practice four types of experiment are customarily considered for the specification of viscoelastic behavior. Linear viscoelastic behavior may be specified in general by either of the equivalent relations:

$$\epsilon(t) = \int_{-\infty}^t \frac{d\sigma(u)}{du} k(t-u) du \quad (1)$$

$$\sigma(t) = \int_{-\infty}^t \frac{d\epsilon(u)}{du} m(t-u) du \quad (2)$$

where $\epsilon(t)$ is a strain and $\sigma(t)$ is a stress at time t , and $k(t)$ and $m(t)$ are time-dependent functions of the dimensions, respectively, of compliance and modulus; these functions represent the viscoelastic properties of the material. It is seen that, if a *unit stress* is applied instantaneously, that is, *stepwise* (and maintained constant thereafter), then the *strain* at time t is given by $k(t)$. Similarly, if a *unit strain* is applied stepwise at zero time, then the *stress* at time t is $m(t)$. The functions $k(t)$ and $m(t)$ define, respectively, the *creep compliance* and *relaxation modulus* functions.

From equation (1) it can be shown that if the stress is varying sinusoidally with time with circular frequency ω , then in the steady state the strain varies sinusoidally with time, and the amplitude of the strain is proportional to the amplitude of the stress. In general, the strain differs in phase from the stress, so under these conditions the ratio of strain to stress can be represented by a frequency-dependent *complex compliance* $k^*(\omega)$. Thus the complex compliance represents in phase and amplitude the strain response, as a function of frequency, to a sinusoidal stress of unit amplitude. Similarly, if the stress is varying sinusoidally with time, then from equation (2) it can be shown that the ratio of stress to strain can be represented by a frequency-dependent complex modulus $m^*(\omega)$.

TABLE I
Symbols and Terminology for Modulus and Compliance

Type of deformation	Remarks	Modulus, m	Compliance, k
Shear		G	J
Bulk (volume)		K	B
Longitudinal (a)	No transverse stress. Strain measured longitudinally	E	D
Longitudinal (b)	No transverse strain. Stress measured longitudinally	M	(N)
Longitudinal (c)	No transverse stress. Strain measured laterally	—	—
Longitudinal (d)	No transverse strain. Stress measured laterally	(λ)	—

The above time- or frequency-dependent moduli and compliances are related to the moduli and compliances of classical elasticity theory. These are given in Table I. It is seen from the table that there are four cross-measures, involving a transverse strain (or stress) and a longitudinal stress (or strain); one of these is the Lamé con-

stant λ . These four measures seem to be of limited use in representation of viscoelastic behavior, and will not be considered further. The compliance N has not yet been used in viscoelasticity; it is introduced here for completeness.

III. RELATIONS BETWEEN STEP-FUNCTION MODULI AND COMPLIANCES

In this report definitions are proposed for three creep compliances, namely, the *shear creep compliance* $J(t)$, the *bulk (or volume) creep compliance* $B(t)$, and the longitudinal creep compliance $D(t)$. The corresponding relaxation moduli are denoted by $G(t)$, $K(t)$, and $E(t)$. The relations between the moduli and compliances of classical elasticity theory are analogous to the relations between the Carson transforms of the time-dependent step-function moduli and compliances given above, where the Carson transform of a creep compliance $k(t)$ is given by:

$$p \int_0^{\infty} e^{-pt} k(t) dt = p\mathcal{L}\{k(t)\}$$

and similarly for the Carson transform of a relaxation modulus. Thus, for example, the relation between a creep compliance $k(t)$ and its associated relaxation modulus $m(t)$ is given by:

$$p\mathcal{L}\{k(t)\} = 1/p\mathcal{L}\{m(t)\}$$

IV. METHODS OF REPRESENTATION OF DYNAMIC BEHAVIOR

Dynamic behavior refers to the response of a material manifesting linear viscoelasticity when the stress and strain vary sinusoidally with time. The strain can be considered as consisting of two components, one in phase with the stress and the other lagging 90° . The ratio of the amplitude of the former component to the amplitude of the stress is the *storage compliance*, $k'(\omega)$. The ratio of the amplitude of the latter component to the amplitude of the stress is the *loss compliance*, $k''(\omega)$. Alternatively, the stress can be considered as consisting of two components, one in phase with the strain, and the other leading by 90° . The ratio of the amplitude of the former component to the amplitude of the strain is the *storage modulus*, $m'(\omega)$;

the ratio of the amplitude of the latter component to the amplitude of the strain is the *loss modulus*, $m''(\omega)$. Thus in general:

$$k^*(\omega) = k'(\omega) - jk''(\omega) \tag{3}$$

and:

$$m^*(\omega) = m'(\omega) + jm''(\omega) \tag{4}$$

The *absolute modulus*, $|m(\omega)|$, is given by:

$$|m(\omega)| = [m'^2(\omega) + m''^2(\omega)]^{1/2} \tag{5}$$

and the *absolute compliance*, $|k(\omega)|$, is given by:

$$|k(\omega)| = [k'^2(\omega) + k''^2(\omega)]^{1/2} \tag{6}$$

The *loss tangent* or *damping*, $\tan \delta$, is given by:

$$\tan \delta = m''(\omega)/m'(\omega) = k''(\omega)/k'(\omega) \tag{7}$$

The quantity $m''(\omega)/\omega$ has the dimensions of viscosity and is generally called the *dynamic viscosity*, $\eta'(\omega)$.

TABLE II
Primary Measurable Quantities
Type of deformation

Quantity	Type of deformation		Longitudinal (a) (no lateral stress)	Longitudinal (b) (no lateral strain)
	Shear	Bulk		
Creep compliance	$J(t)$	$B(t)$	$D(t)$	—
Relaxation modulus	$G(t)$	$K(t)$	$E(t)$	—
Storage compliance	$J'(\omega)$	$B'(\omega)$	$D'(\omega)$	—
Loss compliance	$J''(\omega)$	$B''(\omega)$	$D''(\omega)$	—
Storage modulus	$G'(\omega)$	$K'(\omega)$	$E'(\omega)$	$M'(\omega)$
Loss modulus	$G''(\omega)$	$K''(\omega)$	$E''(\omega)$	$M''(\omega)$
Complex compliance	$J^*(\omega)$	$B^*(\omega)$	$D^*(\omega)$	—
Complex modulus	$G^*(\omega)$	$K^*(\omega)$	$E^*(\omega)$	$M^*(\omega)$
Absolute compliance	$ J(\omega) $	$ B(\omega) $	$ D(\omega) $	—
Absolute modulus	$ G(\omega) $	$ K(\omega) $	$ E(\omega) $	$ M(\omega) $
Loss tangent (damping)	$\tan \delta_s$	$\tan \delta_v$	$\tan \delta_l$	$\tan \delta_p$
Dynamic viscosity	$\eta_s'(\omega)$	$\eta_v'(\omega)$	$\eta_l'(\omega)$	$\eta_p'(\omega)$

In principle, six moduli and compliances can be defined (compare Table I). In practice, three compliances and four moduli are sufficient. The three complex compliances are thus the *complex shear compliance*, $J^*(\omega)$, the *complex bulk compliance*, $B^*(\omega)$, and the *complex lon-*

itudinal compliance, $D^*(\omega)$. The four moduli are, respectively, $G^*(\omega)$, $K^*(\omega)$, $E^*(\omega)$, and $M^*(\omega)$. The relations between the complex moduli and compliances are the same as those existing between the corresponding moduli and compliances of classical elasticity theory. When necessary, values of $\tan \delta$ and $\eta'(\omega)$ corresponding to different moduli may be distinguished by subscripts, thus $\tan \delta_s$, $\tan \delta_e$, $\tan \delta_i$, and $\tan \delta_p$ represent the loss tangents associated, respectively, with $G^*(\omega)$, $K^*(\omega)$, $E^*(\omega)$, and $M^*(\omega)$. Similarly, the corresponding dynamic viscosities may be denoted by $\eta_s'(\omega)$, $\eta_e'(\omega)$, $\eta_i'(\omega)$, and $\eta_p'(\omega)$.

The behavior of a material in step-function or dynamic tests can thus be expressed in terms of the primary measurable quantities given in Table II.

V. LIMITING VALUES

In principle, certain limiting values exist, corresponding to zero and infinite time in step-function experiments, and zero and infinite frequency in dynamic experiments. These limiting values are defined as follows:

$$\left. \begin{aligned} k_g &= \lim_{t \rightarrow 0} k(t) = \lim_{\omega \rightarrow \infty} k'(\omega) \\ m_g &= \lim_{t \rightarrow 0} m(t) = \lim_{\omega \rightarrow \infty} m'(\omega) \\ k_g &= 1/m_g \end{aligned} \right\} \quad (8)$$

The quantities k_g and m_g are called, respectively, the *glass compliance* and *glass modulus*. The quantity defined usually by:

$$\left. \begin{aligned} 1/\eta &= \lim_{t \rightarrow \infty} \{dk(t)/dt\} \\ \eta &= \lim_{\omega \rightarrow 0} \eta'(\omega) \end{aligned} \right\} \quad (9)$$

is called the *viscosity*. Similarly, the quantities k_e and m_e are defined by the relations:

$$\left. \begin{aligned} k_e &= \lim_{t \rightarrow \infty} \{k(t) - t/\eta\} = \lim_{\omega \rightarrow 0} k'(\omega) \\ m_e &= \lim_{t \rightarrow \infty} m(t) = \lim_{\omega \rightarrow 0} m'(\omega) \\ k_e &= 1/m_e \quad \text{if } \eta = \infty \end{aligned} \right\} \quad (10)$$

The quantity k_e may be called the *equilibrium compliance* for a crosslinked material, for which the viscosity is infinite; otherwise, it may be called the *steady-state elastic compliance*. The quantity m_e differs from zero only when the viscosity is infinite, in which case it may be called the *equilibrium modulus*. In practice these limiting values may be experimentally inaccessible; or they may not be determinable experimentally.

VI. EQUIVALENT CONTINUOUS SPECTRA

The creep compliances are related to the complex compliances, and the relaxation moduli are related to the complex moduli through retardation and relaxation spectra, respectively. These spectra may be in principle discrete, continuous, or mixed. Molecular theories of viscoelasticity of amorphous polymers lead to discrete spectra. Since it is not possible to obtain these from experimental data, it is customary to obtain from experimental data the equivalent continuous spectra defined below.

The *retardation spectrum* $L(t)$ is defined by the relationships:

$$\left. \begin{aligned} k(t) - k_g - t/\eta &= \int_{-\infty}^{\infty} (1 - e^{-t/\tau})L(\tau) d \ln \tau \\ k'(\omega) - k_g &= \int_{-\infty}^{\infty} \frac{L(\tau)}{1 + \omega^2\tau^2} d \ln \tau \\ k''(\omega) - 1/\omega\eta &= \int_{-\infty}^{\infty} \frac{\omega\tau L(\tau)}{1 + \omega^2\tau^2} d \ln \tau \end{aligned} \right\} \quad (11)$$

and the *relaxation spectrum* $H(t)$ is defined by the relationships:

$$\left. \begin{aligned} m(t) - m_e &= \int_{-\infty}^{\infty} e^{-t/\tau}H(\tau) d \ln \tau \\ m'(\omega) - m_e &= \int_{-\infty}^{\infty} \frac{\omega^2\tau^2 H(\tau)}{1 + \omega^2\tau^2} d \ln \tau \\ m''(\omega) &= \int_{-\infty}^{\infty} \frac{\omega\tau H(\tau)}{1 + \omega^2\tau^2} d \ln \tau \end{aligned} \right\} \quad (12)$$

When necessary, the retardation and relaxation spectra for shear and for bulk deformation can be distinguished by subscripts, thus we may write $H_s(t)$ and $L_s(t)$ for shear, and $H_v(t)$ and $L_v(t)$ for bulk (volume)

deformation. The terms *relaxation spectrum* and *retardation spectrum* refer to the function over the range of values of t from zero to infinity. The value of $H(t)$ at a specified value of t may be termed the *spectral strength* at time t . The auxiliary quantities, namely, the limiting values and spectra, are listed in Table III. It will be noticed that in this table the viscosity is characterized by a suffix when it is required to distinguish between the longitudinal and shear viscosities; the bulk viscosity of course does not exist.

TABLE III
Auxiliary Quantities

Quantity	Type of deformation		
	Shear	Bulk	Longitudinal (no trans- verse stress)
Glass compliance	J_g	B_g	D_g
Equilibrium compliance	J_e	B_e	D_e
Steady-state elastic compliance			
Glass modulus	G_g	K_g	E_g
Equilibrium modulus	G_e	K_e	E_e
Viscosity	η_s	—	η_l
Retardation spectrum	$L_s(t)$	$L_v(t)$	$L_l(t)$
Relaxation spectrum	$H_s(t)$	$H_v(t)$	$H_l(t)$

VII. DISCUSSION

In this report a minimum of quantities has been defined. It is observed that time- or frequency-dependent quantities are systematically labeled, while limiting quantities are denoted by suffixes. In the system used in this report, the connection between linear viscoelasticity and classical elasticity theory is apparent. For this reason, the more commonly used symbols for elastic moduli (G , K , E , and M) are adopted here. With regard to compliances, J is a generally used symbol, as is also B . The symbol D has been adopted for longitudinal compliance, replacing the previously used symbol F , to which objection was raised. Similarly, the symbol for the "dummy" compliance k replaces the earlier symbol c .

There are some difficulties with regard to nomenclature for the limiting values k_g , m_g , k_e , and m_e . The suffixes zero and infinity do not give rise to confusion in the analogous but not so complete system of

nomenclature for dielectric behavior; with linear viscoelasticity, however, their use is unjustified. The terms "initial" and "instantaneous" modulus and compliance seem uncertain. For amorphous polymers the appropriate modulus and compliance are often obtained as limiting values when the timescale is short enough at any given temperature, the material behaving under these conditions as an organic glass. On this account the terms glass modulus and glass compliance have been chosen, and these quantities have been denoted by m_g and k_g . For other materials, for example, metals, another term would be preferable. It does not seem likely that a term for these limiting values suitable for all categories of materials can be proposed. For the other limiting values, namely, k_e and m_e , the terms "static" and "ultimate" have been suggested. Here again these do not seem to be an improvement over the terms proposed in this report. The letter suffix for the symbol should of course be related to the name adopted for the limiting value. For papers published in languages other than English, it would be desirable that the letter suffixes adopted be associated with the nomenclature for the limiting values in other languages. This desirable state of affairs would seem to be difficult to accomplish.

In the case of nonlinear behavior, the term "viscosity" is sometimes used to denote the ratio of shear stress to rate of shear in general, while the limiting value at very small values of the parameters is denoted by a term such as "Newtonian" viscosity. Since in this report we are dealing only with linear behavior, it seems that such a modifying term is unnecessary here. Objections have been raised to the term "dynamic viscosity" for $m''(\omega)/\omega$. Since this name is well established for a useful quantity, it has been adopted in this report.

Much of the material of previous preliminary versions of this report has been eliminated, in line with the objective of defining a minimum number of quantities in order to promote uniform methods of reporting data. In theoretical treatments it may be convenient to define further functions, for example:

$$\{k(t) - k_a - t/\eta\}/(k_e - k_a)$$

It does not seem either necessary or desirable to standardize a nomenclature for such quantities at this stage.

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